J I A M I N L I

Discrete math ●  APRIL 29, 2016

MTH 153: proof portfolio

# 1| The Product of *n* Odds is Odd

*Claim:* If are odd integers, then their product,

is also odd for all

**(1) Fact :** An integer is odd if and only if it can be expressed as for some integer .

***Proof:*** (by induction)

**➀ *Base Case***: When :

🗸

Since and are odd, there is some integer with and some other integer with . The expression can be rewritten as:

=

*odd*

*even*

Where is an even number because it is a multiple of two, and the integer 1 is odd. So, by definition of an odd number ( the entire expression is odd.

**➁ *Assume:*** the claim holds true for is odd. Then:

**➂ *Prove:*** the claim holds true for

which can be rewritten as

*Odd*

*from claim*

*Odd*

*from ➁*

From our induction hypothesis, we know the is odd, so we can express it in the form where is some integer.

Under the conditions the claim was made under, must also be odd, so we can express it in the form where is some integer.

Rewriting we get

*odd*

*even*

We know that is an even number because it is a multiple of two, and that 1 is an odd number. By definition of an odd number, the sum of an even number and an odd number always gives an odd number. Thus, is an odd number.

***Conclusion:*** By the principle of induction, the claim holds true for all

# 2 | Every 5th Fibonacci number is a Multiple of 5

Recall that the Fibonacci Numbers are defined by the following recursive rule:

For

*Claim:* For all is divisible by 5.

***Proof:*** (by induction)

**➀ *Base Case***: When :

When is divisible by 5 🗸

When is divisible by 5 🗸

**➁ *Assume:*** the claim that for every , is divisible by 5 is true. Then:

**➂ *Prove:*** the claim also holds true for :

is divisible by 5

Recall the recursive rule that defines the Fibonacci Numbers is:

where the value of any Fibonacci number is given by the sum of its two previous terms and .

Expanding , we get . Using the Fibonacci identity, we can rewrite as the sum of its two previous terms

*Continue to expand using the Fibonacci identity*

Combining like terms, we get

*Divisible by 5 (a multiple of 5)*

*Divisible by 5 from ➁*

From our induction hypothesis, we know that is divisible by 5 since 3 times any number divisible by 5 is itself divisible by 5 as well (e.g. 3\*10=30/5).

We also know that is divisible by 5 because any multiple of 5 is divisible by 5.

Thus, since both and are divisible by 5, it must be that their sum is also divisible by 5.

***Conclusion:*** By the principle of induction, the claim holds true for all

# 3 | Fibonacci-many Ways to Tile

*Claim:* The number of ways to tile a -length board is the st Fibonacci Number.

***Proof:*** (by strong induction)

**➀ *Base Case***:

*Number of ways to tile a 2 x n board*

When :

|  |  |
| --- | --- |
|  |  |
|  |  |

|  |  |
| --- | --- |
|  |  |
|  |  |

|  |  |
| --- | --- |
|  |  |
|  |  |

*2 ways*

🗸

When

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |

|  |  |  |
| --- | --- | --- |
|  |  |  |
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| --- | --- | --- |
|  |  |  |
|  |  |  |

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |

*3 ways*

🗸

**➁ *Assume:*** that the claim holds true: the number of ways to tile a board of length is for then

**➂ *Prove:*** that the claim also holds true for a board of length

*2*

First, recall the recursive rule that defines the Fibonacci Numbers

where the value of any Fibonacci number is simply the sum of its two previous terms and .

We consider a board of length and examine the all the possible cases in which we can place tile(s) at the end of the board without tiles hanging off the edge of the boards.

Case 1: Case 2:

|  |  |
| --- | --- |
|  |  |
|  |  |

|  |  |
| --- | --- |
|  |  |
|  |  |

*2*

*2*

Case 1: Suppose we place one vertical tile at the most rightward edge. Then, the white space is represented by the remaining size of the board . We can count how many possible tilings exist for a board. Based on our induction hypothesis, we know that the number of ways to tile a board of length is just .

Case 2: Supposed we place two horizontal tiles to fill the most rightward edge. Then, the white space is represented by the remaining size of the board . We can count how many possible tilings exist for a board. Based on our induction hypothesis, we know that the number of ways to tile a board of length is also .

Adding together the number of possible tilings from case 1 (number of tilings possible for a board) and case 2 (number of tilings possible for a board), we get the number of tilings for a board.

This can be formally written as

Note that this expression is precisely the recursive rule for the Fibonacci Numbers!

***Conclusion:*** By the principle of strong induction, the claim holds true for all

# 4 | Sum of Odds via Induction

*Claim:* For all natural numbers ,

***Proof:*** (by induction)

**➀ *Base Case***: When :

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**➁ *Assume:*** the claim holds true for all natural numbers . Then:

**➂ *Prove:*** the claim also holds true for

*(simplifying RHS)*

Expanding the left hand side:

*(induction hypothesis from ➁)*

*(simplify)*

***Conclusion:*** Since the *LHS* = *RHS*, we have proved that the claim holds for . Thus, by the principle of induction, the claim holds true for all

# 5 | Summing Fibonacci Squares via Induction

Consider the following table ( denotes the Fibonacci Number).

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | 1 | 1 | 2 | 3 | 5 | 8 | 13 | … |
|  | 1 | 1 | 4 | 9 | 25 | 64 | 169 | … |

Notice that: .

Notice that: .

Notice that: .

*Claim:* The squares of the first Fibonacci Numbers sum to the product of the th and st Fibonacci numbers:

Recall that the Fibonacci Numbers are defined by the following recursive rule:

For

***Proof:*** (by induction)

**➀ *Base Case***: When :

🗸

**➁ *Assume:*** the claim holds true for . Then

**➂ *Prove:*** that the claim also holds true for

Using the recursive rule that defines the Fibonacci Numbers, we can rewrite the *RHS* as the sum of its two previous terms .

Expanding the *LHS* of the equation:

*(expand )*

*(simplify)*

*(induction hypothesis from ➁)*

***Conclusion:*** Since the *LHS* = *RHS*, we have proved that the claim holds for . Thus, by the principle of induction, the claim holds for all the first Fibonacci numbers.

# 6 | Dividing a Sum of Squares

*Claim*: For two integers *a* and *b*, assume that . Then at least one of *a* or *b* is even.

***Proof:*** (by contradiction)

→ Assume to the contrary that and *a* is odd and *b* is odd. Since *a* is odd and *b* is odd, by definition there exists an integer *k* such that and similarly an integer *m* such that Then, we have that

Expanding the right hand side:

Note that is an integer due to closure under addition.

Let *j* denote some integer. Then we have that:

where is some integer that is the sum of 2 and some multiple of 4.

Recall that an integer is divisible by 4 only if division by 4 gives a remainder of 0.

Since is divisible by 4, then it follows that the expression cannot be divisible by 4 because it leaves a remainder of 2.

We have reached a contradiction, so we reject our initial assumption and conclude that for two integers *a* and *b,* , at least one of *a* or *b* must be even.

# 7 | The Geometric Mean Never Exceeds the Arithmetic Mean

*Claim:* Assume that *x* and *y* are non-negative. Then the geometric mean never exceeds the arithmetic mean:

***Proof:*** (by contradiction)

→ Assume to the contrary that:

for some .

Rewriting the inequality, we get:

foiling the *RHS..*

factoring the *RHS…*

From our assumption, we know that that and .

We also know that the difference between two non-negative integers always produces an integer due to closure under subtraction.

Therefore, the inequality suggests that for some integer , .

However, by definition, the squaring of some integer always gives an integer

Thus, we have reached a contradiction, so we reject our initial assumption and conclude that

for some .

# 8 | Impossible Combinations

*Claim:* If integers *a* and *b* are not relatively prime, then there exist no integer-coefficients *x* and *y* such that

***Proof*:** (by contradiction)

→ Assume to the contrary that integers *a* and *b* are not relatively prime, and there exists integer-coefficients *x* and *y* such that

If *a* and *b* are not relatively prime, then there exists a prime number *p* such that and .

By the definition of divisibility, there exists an integer *k* such that , and similarly an integer *m* such that .

By substituting and into the equation , we get

Dividing through by *p*, we get .

From our assumption, we know that *x* and *y* are integers. Likewise, from the definition of divisibility, we know that *k* and *m* are integers.

Then, it follows that and are integers due to closure of ℤ under multiplication.

Similarly, is also an integer due to closure of ℤ under addition.

By definition of a prime number, *p* must be strictly greater than 1.

Since , it follows that,.

The equation suggests an integer is equal to a number, , where .

However, it will never be the case that an integer is equivalent to a rational number such that it is between 0 and 1 or -1 and 0.

We have reached a contradiction, therefore we reject our initial assumption and conclude that there exists no integer-coefficients *x* and *y* such that